MULTIPLE COEXISTENCE STATES FOR LOTKA-VOLTERRA COMPETITION MODEL WITH DIFFUSION (Nonlinear Diffusive Systems : Dynamics and Asymptotics)

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MULTIPLE COEXISTENCE STATES FOR LOTKA-VOLTERRA COMPETITION MODEL WITH DIFFUSION

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1. INTRODUCTION

This article is concerned with the following semilinear parabolic system

$$(1.1)\qquad \begin{cases} u_t = k_1\Delta u + u(a-u-cv) & \text{in} \quad \Omega\times(0,\infty),\\ v_t = k_2\Delta v + v(b-du-v) & \text{in} \quad \Omega\times(0,\infty),\\ u=v=0 & \text{on} \quad \partial\Omega\times(0,\infty),\\ u(\cdot,0)=u_0, \quad v(\cdot,0)=v_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, u_0, v_0 are given nonnegative functions in Ω and k_1, k_2, a, b, c, d are positive constants. This system is referred to the Lotka-volterra competition model with diffusion. In (1.1) u and v denote population densities of two competing species. We are interested in positive stationary solutions for (1.1). Such a solution is usually called a coexistence state. The existence, uniqueness and non-uniqueness problem of coexistence states for (1.1) has been studied by many authors (see [1],[2],[3],[4],[8],[9],[10] and the references therein).

The main purpose is to give some remarks on the multiple existence of coexistence states. After rescaling of u and v we are led to the following steady-state problem:

$$(\mathrm{SP})\qquad \begin{cases} \mu\Delta u + u(1-u-cv)=0 & \text{in} \quad \Omega,\\ \nu\Delta v + v(1-du-v)=0 & \text{in} \quad \Omega,\\ u=v=0 & \text{on} \quad \partial\Omega,\\ u\ge 0, \quad v\ge 0 & \text{in} \quad \Omega, \end{cases}$$

where μ, ν, c and d are positive constants. Although non-uniqueness of coexistence states has been discussed in a pretty number of works such as [4], [8], [9], [10], we do not have satisfactory information about explicit conditions for the non-uniqueness. We will give here some sufficient conditions on μ, ν, c, d for the multiple existence of coexistence states in two cases:

$$\begin{array}{ll} (\mathrm{A}) & (c-1)(d-1)<0, cd>1,\\ (\mathrm{B}) & c, d \text{ are sufficiently large.} \end{array}$$

The analysis in the former case is carried out by using the degree theory or local bifurcation theory, while the analysis in the latter case heavily depends on the theory of Dancer and Du [5].

In Section 2 we will give some preliminary results on the existence of coexistence states for (SP). Multiple existence for case (A) is discussed in Section 3. In Section 4

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we get some multiplicity results of (SP) from the analysis of suitable limit problems with $c, d \to \infty$ in (SP).

2. Preliminaries

We begin with the following auxiliary problem for a semilinear elliptic equation:

$$\begin{cases} \mu\Delta w + w(1-w) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is well known that (2.1) has a unique positive solution φ_μ if and only if $0 < \mu < \sigma^* := 1/\lambda_1$, where λ_1 is the least eigenvalue for $-\Delta w = \lambda w$ in Ω with $w = 0$ on $\partial\Omega$. Moreover, it is also possible to show the following result (see, e.g., [10]).

Lemma 2.1. (i) *If $0 < \mu < \sigma^*$, then there exists a unique positive solution φ_μ of* (2.1) *such that $\varphi_\mu(x)$ is strictly decreasing with respect to μ for every $x \in \Omega$.*
(ii) *$\mu \to \varphi_\mu$ is a C^1-mapping from $(0, \sigma^*)$ to $C_0(\overline{\Omega})$, where $C_0(\overline{\Omega})$ denotes the space of all continuous functions u in $\overline{\Omega}$ such that u vanishes on $\partial\Omega$.*
(iii) $\lim_{\mu\to\sigma^*} \varphi_\mu = 0$ *uniformly in Ω. More precisely,*

$$\varphi_\mu = \frac{\lambda_1(\sigma^* - \mu)}{m_0}\varphi^* + o(\sigma^* - \mu) \qquad \text{as} \quad \sigma^* - \mu \to 0, \tag{2.2}$$

where $m_0 = \int_\Omega \varphi^*(x)^3 dx$.
(iv) *For any compact subset F in Ω,* $\lim_{\mu\to 0} \varphi_\mu = 1$ *uniformly in F.*

Lemma 2.1 assures that (SP) has no coexistence states for $\mu \ge \sigma^*$ or $\nu \ge \sigma^*$; so we assume

$$0 < \mu < \sigma^* \qquad \text{and} \qquad 0 < \nu < \sigma^*$$

in the sequel.

Define

$$f(\mu) = \sup\left\{ \int_\Omega (1 - d\varphi_\mu)w^2 dx \,/ \|\nabla w\|^2; \;\; w \in H_0^1(\Omega), w \ne 0 \right\}, \tag{2.3}$$

where $\|\cdot\|$ denotes $L^2(\Omega)-$norm.

Lemma 2.2. *If f is defined by* (2.3), *then it has the following properties.*
(i) *f is a strictly increasing function of class C^1 in $(0, \sigma^*)$.*
(ii) $\lim_{\mu\to\sigma^*} f(\mu) = \sigma^*$ *and* $\lim_{\mu\to\sigma^*} f'(\mu) = d$.
(iii) $\lim_{\mu\to 0} f(\mu) = (1-d)^+\sigma^*$.

Proof. In order to prove (i) we will employ the argument in the proof of [16, Lemma 3.4]. We first observe that the supremum in (2.3) is attained by a positive function

$w_\mu \in H_0^1(\Omega)$, which is normalized with $\|\nabla w_\mu\| = 1$. It follows from the definition that

$$
(2.4)\qquad
\begin{aligned}
f(\mu+h) &= \int_\Omega (1 - d\varphi_{\mu+h})w_{\mu+h}^2 dx \geq \int_\Omega (1 - d\varphi_{\mu+h})w_\mu^2 dx \\
&= \int_\Omega (1 - d\varphi_\mu)w_\mu^2 dx + d\int_\Omega (\varphi_\mu - \varphi_{\mu+h})w_\mu^2 dx \\
&= f(\mu) + d\int_\Omega (\varphi_\mu - \varphi_{\mu+h})w_\mu^2 dx.
\end{aligned}
$$

Since a similar inequality to (2.4) holds true if μ and $\mu + h$ are exchanged, one can derive

$$
(2.5)\qquad |f(\mu+h) - f(\mu)| \leq C\|\varphi_{\mu+h} - \varphi_\mu\|_\infty
$$

for some $C > 0$, where $\|\cdot\|_\infty$ denotes the supremum norm. Thus (2.5), together with Lemma 2.1, implies the Lipschitz continuity of f with respect to μ. It is easy to see $\lim_{\mu\to\sigma^*} f(\mu) = \sigma^*$ from (iii) of Lemma 2.1 because $w_\mu \to w^*$ in $H_0^1(\Omega)$, where w^* satisfies $\mu^*\Delta w^* + w^* = 0$ in Ω and $\|\nabla w^*\| = 1$.

The Lipschitz continuity also means that $f(\mu)$ is differentiable for almost every $\mu \in (0, \sigma^*)$. Making use of (2.4) we divide $f(\mu+h) - f(\mu)$ by $h > 0$ ($h < 0$) and let $h \to 0$; then

$$
(2.6)\qquad f'(\mu) = -d\int_\Omega \frac{\partial \varphi_\mu}{\partial \mu} w_\mu^2 dx
$$

for almost every $\mu \in (0, \sigma^*)$. By Lemma 2.1, the right-hand side of (2.6) is continuous in $\mu \in (0, \sigma^*]$; so that (2.6) is valid for every $\mu \in (0, \sigma^*]$. Clearly, (2.6) together with (2.2) yields $f'(\mu) > 0$ and

$$
\lim_{\mu\to\sigma^*} f'(\mu) = \frac{\lambda_1 d}{m_0}\int_\Omega \frac{(\varphi^*)^3}{\|\nabla\varphi^*\|^2} dx = d,
$$

where we have used $w^* = \varphi^*/\|\nabla\varphi^*\|$ and $\|\nabla\varphi^*\|^2 = \lambda_1$.

It remains to show (iii). From the monotonicity in (i) there exists a limit of $f(\mu)$ as $\mu \to 0$; so we put

$$
\lim_{\mu\to 0} f(\mu) = \nu^*.
$$

Since $\varphi_\mu \leq 1$ in Ω, it is easy to see $f(\mu) \geq (1-d)\|w\|^2/\|\nabla w\|^2$ for all $w \in H_0^1(\Omega)$ and $\mu \in (0, \sigma^*)$; so that, in view of $\sup\{\|w\|^2/\|\nabla w\|^2; w \in H_0^1(\Omega)\} = \sigma^*$, we get

$$
\nu^* \geq (1-d)\sigma^*.
$$

Moreover, even if the set $\{x \in \Omega; d\varphi_\mu(x) > 1\}$ is non-empty, we can choose a suitable function $w \in H_0^1(\Omega)$ such that $\int_\Omega (1 - d\varphi_\mu)w^2 dx > 0$. This fact means $f(\mu) > 0$ for every $\mu > 0$. Therefore,

$$
(2.7)\qquad \nu^* \geq \max\{(1-d)\sigma^*, 0\}.
$$

To prove the reverse inequality, we use a family $\{w_\mu\}$ again. Since $\|\nabla w_\mu\| = 1$, it follows from Rellich's theorem that there exists a sequence $\{\mu_n\} \downarrow 0$ such that $w_n = w_{\mu_n} (n = 1, 2, 3, \cdots)$ satisfy

$$
\begin{aligned}
&\lim_{n\to\infty} w_n = w_\infty && \text{strongly in } L^2(\Omega), \\
&\lim_{n\to\infty} \nabla w_n = \nabla w_\infty && \text{weakly in } L^2(\Omega),
\end{aligned}
$$

for some $w_\infty \in H_0^1(\Omega)$. Note that $\|\nabla w_\infty\|^2 \le 1$. As in the proof of [14, Lemma A.1], one can prove

$$\nu^* = \lim_{n\to\infty} \int_\Omega (1 - d\varphi_{\mu_n}) w_n^2 dx = (1-d)\|w_\infty\|^2 \tag{2.8}$$

by Lemma 2.1 and Lebesgue's dominated convergence theorem. If $0 < d < 1$, then (2.7) and (2.8) imply $w_\infty \neq 0$; so that it follows from (2.8) that

$$\nu^* \le \frac{(1-d)\|w_\infty\|^2}{\|\nabla w_\infty\|^2} \le (1-d)\sigma^*,$$

which, together with (2.7), yields the assertion. In case $d > 1$, (2.7) and (2.8) imply $w_\infty = 0$, which shows $\nu^* = 0$. Thus we complete the proof. □

Similarly, if we define

$$g(\nu) = \sup\left\{ \int_\Omega (1 - c\varphi_\nu) w^2 dx \,/ \|\nabla w\|^2;\ \ w \in H_0^1(\Omega), w \neq 0 \right\}, \tag{2.9}$$

then we can show an analogous result for g.

Lemma 2.3. *If g is defined by (2.9), then it possesses the following properties.*
(i) *g is a strictly increasing function of class C^1 in $(0, \sigma^*)$.*
(ii) *$\lim_{\nu\to\sigma^*} g(\nu) = \sigma^*$ and $\lim_{\nu\to\sigma^*} g'(\nu) = c$.*
(iii) *$\lim_{\nu\to 0} g(\nu) = (1-c)^+\sigma^*$.*

We are now ready to state the existence result, whichi is essentially due to Dancer [3] or Blat-Brown [1]. See also [16, Theorem 3.6], in which the idea of the proof can be found.

Theorem 2.1. *Define*

$$\Gamma^+ = \{(\mu,\nu) \in (0,\sigma^*) \times (0,\sigma^*); \nu < f(\mu) \quad \text{and} \quad \mu < g(\nu)\},$$
$$\Gamma^- = \{(\mu,\nu) \in (0,\sigma^*) \times (0,\sigma^*); \nu > f(\mu) \quad \text{and} \quad \mu > g(\nu)\},$$

and set $\Gamma = \Gamma^+ \cup \Gamma^-$. If $(\mu,\nu) \in \Gamma$, then (SP) has at least one coexistence state.

In $\mu\nu$-plane draw two curves s_1 and s_2 defined by

$$s_1 : \nu = f(\mu) \qquad \text{and} \qquad s_2 : \mu = g(\nu);$$

so that Γ is a region surrounded by s_1 and s_2. By Lemmas 2.2 and 2.3, Γ^+ is non-empty if $cd \le 1, (c,d) \neq (1,1)$ and Γ^- is non-empty if $cd > 1$. Moreover, if $cd > 1$ and $(c-1)(d-1) < 0$, then both Γ^+ and Γ^- are non-empty; in particular, s_1 and s_2 meet at a point except for (σ^*, σ^*).

Remark 2.1. Theorem 2.1 implies that (SP) admits at least one coexistence state for $(\mu,\nu) \in \Gamma$. However, we do not have much information on the uniqueness and non-uniqueness of coexistence states of (SP) except for $\mu = \nu$. In the special case $\mu = \nu$, Cosner-Lazer [2] have proved that, if $c < 1$ and $d < 1$, then (SP) admits a unique coexistence state and that, if $c = d = 1$, then there exists a continuum of coexistence states for (SP). Moreover, Gui-Lou [10] have shown that, if $c > 1$ and $d > 1$, then the situation becomes more complicate and the uniqueness and non-uniqueness results depend on the size of diffusion coefficients $\mu = \nu$.

3. Multiple existence in case (A)

In this section we will give some conditions under which (SP) has at least two coexistence states in case

$$\text{(A)} \qquad (c-1)(d-1) < 0 \quad \text{and} \quad d > 1.$$

We will review Theorem 1.1 from the view-point of bifurcation theory. Let $\mu \in (0, \sigma^*)$ be fixed and set $\nu^* = f(\mu)$. We construct bifurcating solutions, which emerge from $\{\varphi_\mu, 0\}$ at $\nu = \nu^*$, by regarding ν as a parameter and making use of the local bifurcation theory. Define a positive function Ψ_μ by

$$\text{(3.1)} \qquad \begin{cases} \nu^* \Delta \Psi_\mu + (1 - d\varphi_\mu)\Psi_\mu = 0 & \text{in } \Omega, \\ \Psi_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

and determine Φ_μ by

$$\text{(3.2)} \qquad \begin{cases} \mu \Delta \Phi_\mu + (1 - 2\varphi_\mu)\Phi_\mu = c\varphi_\mu \Psi_\mu & \text{in } \Omega, \\ \Phi_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $(-\mu\Delta + (2\varphi_\mu - 1)I)^{-1}$ is a strongly order-preserving operator and Ψ_μ is positive in Ω, one can see $\Phi_\mu < 0$ in Ω from (3.2). We normalize Φ_μ and Ψ_μ so that they satisfy $\|\Phi_\mu\|^2 + \|\Psi_\mu\|^2 = 1$. If a new parameter ϵ is introduced, coexistence states $(u, v) = (u(\epsilon), v(\epsilon))$ of (SP) with $\nu = \nu(\epsilon)$, which bifurcate from $\{\varphi_\mu, 0\}$ at $\nu = \nu^*$, can be expressed as

$$\text{(3.3)} \qquad \begin{cases} u(\epsilon) = \varphi_\mu + \epsilon\Phi_\mu + o(\epsilon), \\ v(\epsilon) = \epsilon\Psi_\mu + o(\epsilon), \\ \nu(\epsilon) = \nu^* + \nu_1(\mu)\epsilon + o(\epsilon), \end{cases}$$

for $0 < \epsilon < \epsilon_0$ with some ϵ_0. Recall $\Phi_\mu < 0$ and $\Psi_\mu > 0$ in Ω in (3.3); so the sign of $\nu_1(\mu)$ determines the direction of bifurcation with respect to ν. Here we note the following lemma.

Lemma 3.1. *Let $\mu \in (0, \sigma^*)$ be fixed and let $\{u(\epsilon), v(\epsilon)\}$ be a family of coexistence states of* (SP) *with $\nu = \nu(\epsilon)$ of the form* (3.3). *Then it holds that*

$$\nu_1(\mu)\|\nabla\Psi_\mu\|^2 = -\int_\Omega \Psi_\mu^2 (d\Phi_\mu + \Psi_\mu)dx.$$

Proof. Substitution of (3.3) into the second equation of (SP) yields

$$\text{(3.4)} \quad \nu^*\Delta V(\epsilon) + (1 - d\varphi_\mu V(\epsilon) + \epsilon\nu_1 \Delta\Psi_\mu - \epsilon\Psi_\mu(d\Phi_\mu + \Psi_\mu) = o(\epsilon) \quad \text{in } \Omega \quad \text{as } \epsilon \to 0$$

with some $V(\epsilon) \subset C_0(\overline{\Omega})$ satisfying $\int_\Omega V(\epsilon)\Psi dx = 0$. Taking L^2-inner product of (3.4) with Ψ_μ leads us to

$$\epsilon\nu_1\|\nabla\Psi_\mu\|^2 + \epsilon\int_\Omega \Psi_\mu^2(d\Phi_\mu + \Psi_\mu)dx = o(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

(use (3.1)). Hence dividing the above identity by ϵ and letting $\epsilon \to 0$ we get the conclusion. $\square$

Remark 3.1. Lemma 3.1 tells us the sign of $\mu_1(\nu)$ and, therefore, the direction of the bifurcation of coexistence states from $\{\varphi_\mu, 0\}$ at $\nu = f(\mu)$. The bifurcation is supercritical (resp. subcritical) if $\mu_1(\nu) > 0$ (resp. $\mu_1(\nu) < 0$). Moreover, we can also study the stability or instability of the bifurcating solutions. Indeed, $\{u(\epsilon), v(\epsilon)\}$ is asymptotically stable (resp. unstable) if $\mu_1(\nu) < 0$ (resp. $\mu_1(\nu) > 0$).

Theorem 3.1. *Let (μ_0, ν_0) be an intersection point of s_1 and s_2 curves. If $\nu_1(\mu_0) \neq 0$, then* (SP) *admits at least two coexistence states for (μ, ν) in an open set Λ near (μ_0, ν_0).*

The proof of Theorem 3.1 can be accomplished by using the local bifurcation theory or the degree theory (see, e.g., Yamada [16]).

We will review Theorem 3.1 from the point of the global bifurcation theory. In [1] Blat and Brown have shown that, for fixed $\mu \in (0, \sigma^*)$, there exists a branch of coexistence states for (SP) such that the branch bifurcating from $\{\varphi_\mu, 0\}$ at $(\mu, f(\mu)) \in s_1$ connects with $\{0, \varphi_{\nu_*}\}$ at $(\mu, \nu_*) \in s_2$ satisfying $g(\nu_*) = \mu$ (see also [5] or [9]).

Now let (μ_0, ν_0) be an intersection point of s_1 and s_2 and assume $\nu_1(\mu_0) \neq 0$. Theorem 3.1 means that each branch of coexistence states has a bending point in the bifurcation diagram provided that μ lies in a suitable range $I(\mu_0)$ near $\mu = \mu_0$. For each $\mu \in I(\mu_0)$, let the branch possess a bending point at $\nu = \overline{\nu}(\mu) > f(\mu)$ (resp. $\underline{\nu}(\mu) < f(\mu)$) in the case of supercritical bifurcation $\nu_1(\mu) > 0$ (resp. subcritical bifurcation $\nu_1(\mu) < 0$). Suppose $\nu_1(\mu) > 0$ for $\mu \in I(\nu_0)$. Then (SP) has at least two coexistence states if $\nu \in (f(\mu), \overline{\nu}(\mu))$. Analogous results are also valid for $\nu_1(\mu) < 0$.

We give a numerical example carried out by Professor Etsushi Nakaguchi (Osaka University). For $\Omega = (0, 1)$ with $N = 1$, he has studied

$$
(3.5) \qquad \begin{cases} \mu u'' + u(1 - u - cv) = 0 & \text{in } (0,1), \\ \nu v'' + v(1 - du - v) = 0 & \text{in } (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, & \\ u \geq 0, \quad v \geq 0 & \text{in } (0,1), \end{cases}
$$

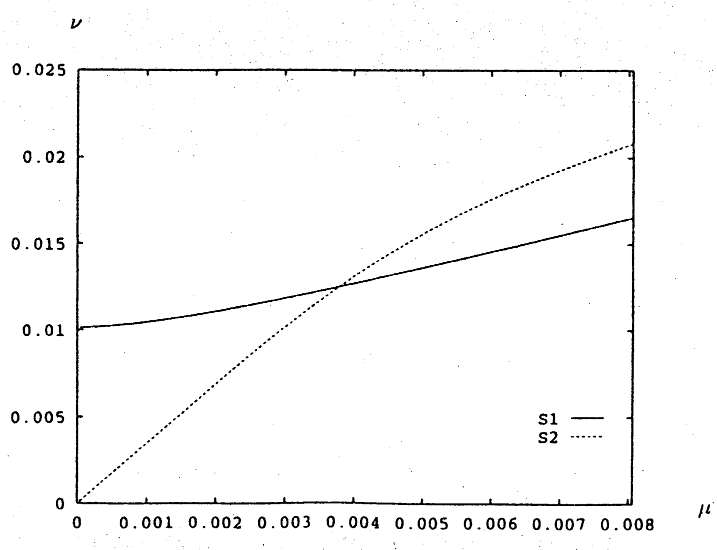


FIGURE 1. Local view of s_1 and s_2.

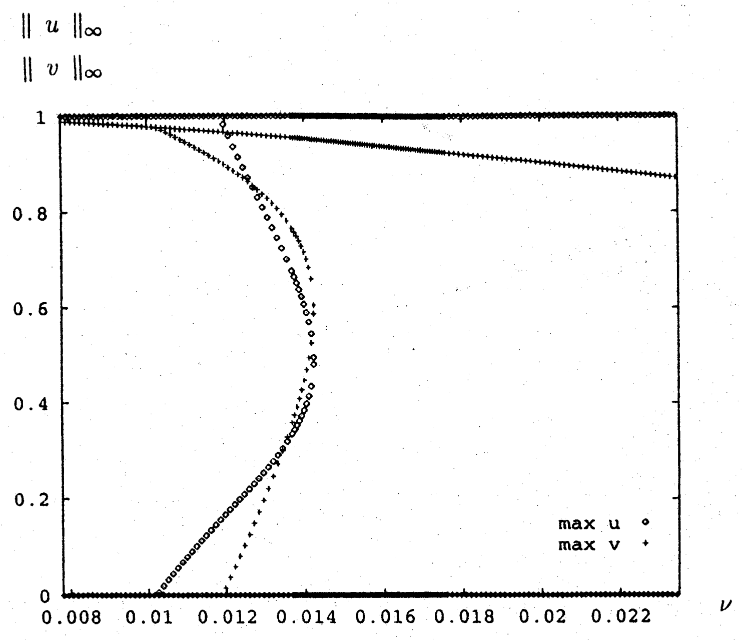


FIGURE 2. Bifurcation diagram of coexistence states for $\mu = 0.003$. There exists a branch of coexistence states emerging from $\{\varphi_\mu, 0\}$ at $\nu = 0.0120$ (s_1 curve) and connecting to $\{0, \varphi_\nu\}$ at $\nu = 0.0102$ (s_2 curve). This branch has a turning point at $\nu = 0.0142$.

with $c = 1.2$ and $d = 0.9$, which satisfy condition (A). So two curves s_1 and s_2 meet at a point $(\mu_0, \nu_0) = (0.0039, 0.013)$ as in Figure 1. For $\mu = 0.03$, Figure 2 shows that the bifurcation of coexistence states at $\nu = 0.0120$ is supercritical and that this branch has a bending point at $\nu = 0.0142$. Therefore, if $\nu \in (0.0120, 0.0142)$, then (3.5) admits two coexistence states. In Figure 3, we are studying the stability properties of semitrivial solutions and positive solutions. The vertical axis denotes the position of the principal eigenvalue for the linearized operator.

Remark 3.2. Let $\nu_1(\mu_0) = 0$. According to Li and Logan [12], (SP) admits a continuum of coexistence states or a coexistence state for $(\mu, \nu) = (\mu_0, \nu_0)$. In the former case, the set Λ in Theorem 3.1, where non-uniqueness result holds true, may be identical with a single point $\{(\mu_0, \nu_0)\}$.

Remark 3.3. Let $(\mu, \nu) \in (0, \sigma^*) \times (0, \sigma^*)$ be fixed. One can show that s_1 curve moves downward as d becomes larger. The situation is similar with respect to s_2 curve; so that (μ, ν) eventually enter Γ^- if c, d become sufficiently large. Therefore, Theorem 2.1 tells us that (SP) has a coexistence state for such c, d. In Section 4 we will show that (SP) admits a finitely many number of coexistence states if μ, ν are small and c, d are sufficiently large.

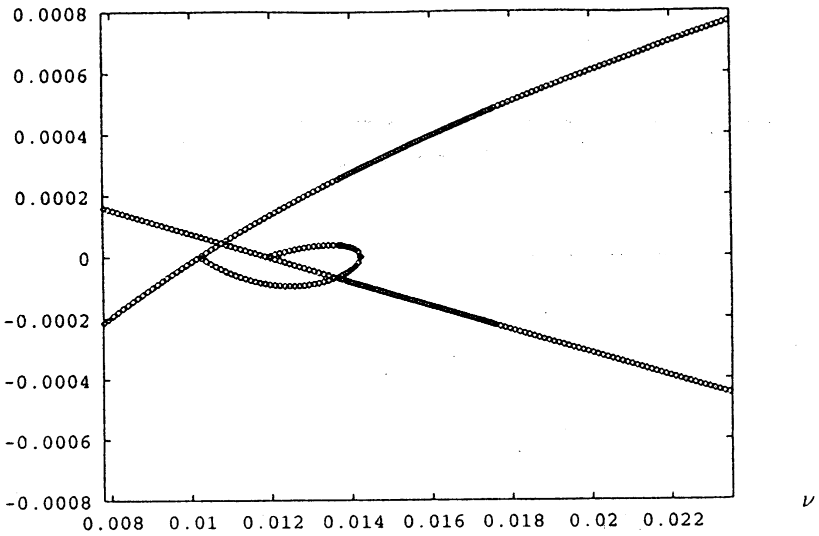


FIGURE 3. Stability of steady-states. The vertical axis indicates the principal eigenvalue for the corresponding linearized operator. The co-existence states bifurcating from $\{\varphi_\mu, 0\}$ are unstable for a certain range of ν, while those bifurcating from $\{0, \varphi_\nu\}$ are asymptotically stable for the same range of ν.

4. MULTIPLE EXISTENCE IN CASE (B)

The analysis in this section employs the theory of Dancer and Du [5], who discuss (SP) for sufficiently large interactions. According to their theory, if $c/d \to \alpha \in (0, \infty)$ as $c, d \to \infty$, then there is a close relationship between (SP) and the following limit problem

$$
\begin{cases} \Delta w + \dfrac{w^+}{\mu}\left(1 - \dfrac{w^+}{\mu}\right) + \dfrac{w^-}{\nu}\left(1 + \dfrac{w^-}{\nu\alpha}\right) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

where $w^+ = \max\{w, 0\}$ and $w^- = \min\{w, 0\}$. Indeed, Dancer and Du have established the following result.

Theorem 4.1. [5, Theorem 2.2] *Assume that $c_n, d_n \to +\infty$ with $c_n/d_n \to \alpha$ as $n \to +\infty$. Let $\{u_n, v_n\}$ be positive solutions of* (SP) *with $(c, d) = (c_n, d_n)$ such that $c_n\|v_n\|_\infty \to +\infty$ and $d_n\|u_n\|_\infty \to +\infty$ as $n \to +\infty$, where $\|\cdot\|_\infty$ denotes the usual supremum norm in Ω. Moreover, assume that $w = 0$ is a unique solution of*

$$
\begin{cases} \Delta w + \dfrac{1}{\mu}w^+ + \dfrac{1}{\nu}w^- = 0 & \textit{in } \Omega, \\ w = 0 & \textit{on } \partial\Omega. \end{cases}
$$

Then there exists a subsequence of $\{u_n, v_n\}$ which converges in $L^2(\Omega) \times L^2(\Omega)$ to $\{w_0^+/\mu, -w_0^-/\nu\alpha\}$ for a solution w_0 of (4.1) *which changes sign in Ω.*

Dancer and Du have also shown that (4.1) gives some useful information on coexistence states of (SP) for sufficiently large c, d in the following sense:

Theorem 4.2. [5, Theorem 3.3] *Let w_0 be an isolated solution of (4.1) such that w_0 changes sign in Ω and index of $w_0 \neq 0$. Then for any $\epsilon > 0$ there exist positive constants M large and δ small such that for every c, d satisfying*

$$c \geq M \qquad \text{and} \qquad |\frac{c}{d} - \alpha| < \delta,$$

(SP) *admits a positive solution $\{u, v\}$ such that*

$$\| u - \frac{w^+}{\mu} \| < \epsilon \qquad \text{and} \qquad \| v + \frac{w^-}{\nu\alpha} \| < \epsilon.$$

Here the index of w_0 means the fixed point index

$$\mathrm{index}_{C_0^1(\Omega)}(A, w_0)$$

with

$$Aw = (-\Delta)^{-1}\left(\frac{w^+}{\mu}\left(1 - \frac{w^+}{\mu}\right) + \frac{w^-}{\nu}\left(1 + \frac{w^-}{\nu\alpha}\right)\right). \tag{4.2}$$

Remark 4.1. In the case when $c/d \to +\infty$ as $c, d \to \infty$, analogous theorems as Theorems 4.1 and 4.2 hold true with (4.1) replaced by

$$\begin{cases} \Delta w + \dfrac{w^+}{\mu}\left(1 - \dfrac{w^+}{\mu}\right) + \dfrac{w^-}{\nu} = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

and (4.2) replaced by

$$Aw = (-\Delta)^{-1}\left(\frac{w^+}{\mu}\left(1 - \frac{w^+}{\mu}\right) + \frac{w^-}{\nu}\right).$$

See [5, Theorems 2.3 and 3.4].

If we can show that (4.1) or (4.3) has many isolated solutions which change signs and have non-zero indices, then theorem 4.2 and Remark 4.1 assure that (SP) admits many coexistence states for sufficiently large c, d.

In what follows, we study (4.1) in a special case $\Omega = (0, 1)$ with $N = 1$:

$$\begin{cases} w'' + h(w) = 0 & \text{in } (0, 1), \\ w(0) = w(1) = 0, \end{cases} \tag{4.4}$$

where

$$h(w) = \frac{w^+}{\mu}\left(1 - \frac{w^+}{\mu}\right) + \frac{w^-}{\nu}\left(1 + \frac{w^-}{\nu\alpha}\right).$$

Since (4.4) is a boundary value problem for an ordinary differential equation, it is possible to get a complete information on the structure of solutions by the standard phase plane analysis. See the work of Dancer, Hilhorst, Mimura and Peletier [7], where a similar problem has been discussed.

In a master's thesis of my graduate student T. Hirose [11] a complete result is obtained for the structure of non-trivial solutions for (4.4). We will summarize his

existence results. Let $w_{k,+}$ (resp. $w_{k,-}$) denote a solution of (4.4) which changes sign $k-$times in $(0,1)$ with positive (resp. negative) first derivative at $x=0$. Then one can show the following result:
(i) there exists a unique solution $w_{2k,+}$ of (4.4) if and only if $(k+1)\sqrt{\mu}+k\sqrt{\nu}<1/\pi$,
(ii) there exists a unique solution $w_{2k,-}$ of (4.4) if and only if $k\sqrt{\mu}+(k+1)\sqrt{\nu}<1/\pi$,
(iii) there exists a unique pair of solutions $w_{2k-1,\pm}$ of (4.4) if and only if $k\sqrt{\mu}+k\sqrt{\nu}<1/\pi$.

These results help us to determine the set $W := \{w \in C^2[0,1]; w \text{ is a solution of } (4.4)\}$. We define the following sets in $\mu\nu$-plane:

$$\begin{aligned}
D_k^1 &= \left\{(\mu,\nu); k(\sqrt{\mu}+\sqrt{\nu})<\frac{1}{\pi}, (k+1)\sqrt{\mu}+k\sqrt{\nu}\geq\frac{1}{\pi}, k\sqrt{\mu}+(k+1)\sqrt{\nu}\geq\frac{1}{\pi}\right\},\\
D_k^2 &= \Bigg\{(\mu,\nu); (k+1)(\sqrt{\mu}+\sqrt{\nu})\geq\frac{1}{\pi}, (k+1)\sqrt{\mu}+k\sqrt{\nu}<\frac{1}{\pi},\\
&\qquad\qquad k\sqrt{\mu}+(k+1)\sqrt{\nu}<\frac{1}{\pi}\Bigg\},\\
D_k^3 &= \left\{(\mu,\nu); (k+1)\sqrt{\mu}+k\sqrt{\nu}\geq\frac{1}{\pi}, k\sqrt{\mu}+(k+1)\sqrt{\nu}<\frac{1}{\pi}\right\},\\
D_k^4 &= \left\{(\mu,\nu); (k+1)\sqrt{\mu}+k\sqrt{\nu}<\frac{1}{\pi}, k\sqrt{\mu}+(k+1)\sqrt{\nu}\geq\frac{1}{\pi}\right\},
\end{aligned}$$

where k is a non-negative integer. Making use of the above results (i), (ii) and (iii) one can show

Lemma 4.1. *Let $(\mu,\nu)\in(0,\sigma^*)\times(0,\sigma^*)$. Then it holds that*

$$W=\begin{cases}
\{0,\ w_{0,\pm}\} & \text{if } (\mu,\nu)\in D_0^2,\\
\{0,\ w_{0,\pm},\ w_{1,\pm},\ \cdots,\ w_{2k-1,\pm}\} & \text{if } (\mu,\nu)\in D_k^1,\\
\{0,\ w_{0,\pm},\ w_{1,\pm},\ \cdots,\ w_{2k,\pm}\} & \text{if } (\mu,\nu)\in D_k^2,\\
\{0,\ w_{0,\pm},\ w_{1,\pm},\ \cdots,\ w_{2k-1,\pm},\ w_{2k,-}\} & \text{if } (\mu,\nu)\in D_k^3,\\
\{0,\ w_{0,\pm},\ w_{1,\pm},\ \cdots,\ w_{2k-1,\pm},\ w_{2k,+}\} & \text{if } (\mu,\nu)\in D_k^4,
\end{cases}$$

for $k=1,2,3,\cdots$. In particular, every element of W is isolated.

Remark 4.2. One-dimensional version of (4.2) is given by

$$\text{(4.5)}\qquad \begin{cases} w''+g(w)=0 & \text{in } (0,1),\\ w(0)=w(1)=0, \end{cases}$$

with

$$g(w)=\frac{w^+}{\mu}\left(1-\frac{w^+}{\mu}\right)+\frac{w^-}{\nu}.$$

The same result as Lemma 4.1 also holds true for (4.5).

Moreover, Hirose [11] has shown that every non-trivial solution of (4.4) or (4.5) has non-zero index. Indeed, the following theorem holds true.

Theorem 4.3. *Let $w_{m,\pm}, m = 0, 1, 2, \cdots$, be any solution of* (4.4) *or* (4.5). *Then it holds that*

$$\text{index of } w_{m,\pm} = (-1)^m \qquad \text{for} \qquad m = 0, 1, 2, \cdots .$$

Remark 4.3. In (4.4) and (4.5), reaction terms are not smooth in case $\mu \neq \nu$; so that A defined by (4.3) is not of class C^1. Hence one cannot directly aplly the index formula to get the assertion of Theorem 4.3. To prove this theorem we need some devices based on the homotopy invariance of the degree.

We can see from Lemma 4.1 and Remark that (4.4) or (4.5) admits a sign-changing solution if and only if $\sqrt{\mu} + \sqrt{\nu} < 1/\pi$. Each sign-changing solution satisfies the assumptions of Theorem 4.2 by virtue of Lemma 4.1 and Theorem 4.3. Therefore, one can apply Theorem 4.2 for each sign-changing solution to get the corresponding coexistence state for large interactions (see also the work of Dancer and Guo [6]).

Theorem 4.4. *Suppose that $(\mu, \nu) \in \bigcup_{i=1}^{4} D_k^i$ for $k \in \mathrm{N}$. Then there exist large numbers c^* and d^* such that for every $c \geq c^*$ and $d \geq d^*$ the following properties hold true:*
(i) *if $(\mu, \nu) \in D_k^1$, then* (SP) (*or* (3.5)) *admits at least* $(4k - 2)$ *coexistence states,*
(ii) *if $(\mu, \nu) \in D_k^2$, then* (SP) (*or* (3.5)) *admits at least* $4k$ *coexistence states,*
(iii) *if $(\mu, \nu) \in D_k^3 \cup D_k^4$, then* (SP) (*or* (3.5)) *admits at least* $(4k-1)$ *coexistence states.*

Remark 4.4. Theorem 4.4 says that, if (4.4) or (4.5) has a sign-changing solution, then (SP) has a coexistence state which is very close to such a solution (in a certain sense) with respect to $L^2(\Omega)$-norm if c, d are sufficiently large. If we use stability results due to Dancer and Guo [6], we can get more information on the instability of the above coexistence state. Indeed, the comparison method enables us to show that every changing-sign solution w_0 of (4.4) or (4.5) is unstable as a stationary solution of the natural corresponding parabolic equation. Therefore, if the non-degeneracy of w_0 is established, then it becomes linearly unstable; so that Theorem 2.2 in [6] implies that the coexistence state of (SP) associated with w_0 is unstable when c, d are sufficiently large.

We can also see that profiles of these coexistence states are very similar to those of limit-solutions given by sign-changing solutions. In this connection, it should be noted that the following theorem holds true. See [11].
Theorem 4.5. *Let $\{u, v\}$ be any coexistence state of* (SP).
(i) *u and v have a finite number of local maximum points in* $(0, 1)$.
(ii) *Let $x_1 < x_2 < \cdots < x_m$ be local maximum points of u in $(0,1)$ and let $y_1 < y_2 < \cdots < y_n$ be local maximum points of v in $(0, 1)$. Then $|m - n| \leq 1$.*
(iii) *Local maximum points of u and those of v appear alternately.*

The proof of Theorem 4.5 can be accomplished along the idea used by Nakashima [13]. We can also show the following result.

Theorem 4.6. *Let $\{c_n, d_n\}$ satisfy $c_n \to \infty$ and $d_n \to \infty$ with $c_n/d_n \to \alpha$ as $n \to \infty$ and let $\{u_n, v_n\}$ be a coexistence state of* (SP)(*or* (3.5) *such that*

$$\{u_n, v_n\} \to \{\frac{1}{\mu}(w_k)^+, -\frac{1}{\nu\alpha}(w_k)^-\} \qquad in \quad L^2(\Omega) \quad as \quad n \to \infty,$$

for some $k \in \mathrm{N}$, where w_k is a changing-sign solution of (4.4). *Then for any $\epsilon > 0$ there exixts a sufficiently large n^* such that, for any $n \geq n^*$*

$$\begin{aligned} &\textit{the number of local maximum points of } u_n \textit{ in } (\epsilon, 1-\epsilon) \\ = \; &\textit{the number of local maximum points of } (w_k)^+ \textit{ in } (0,1) \end{aligned}$$

and

$$\begin{aligned} &\textit{the number of local maximum points of } v_n \textit{ in } (\epsilon, 1-\epsilon) \\ = \; &\textit{the number of local minimum points of } (w_k)^- \textit{ in } (0,1). \end{aligned}$$

Here we will give some numerical examples accomplished by Hirose for the following system

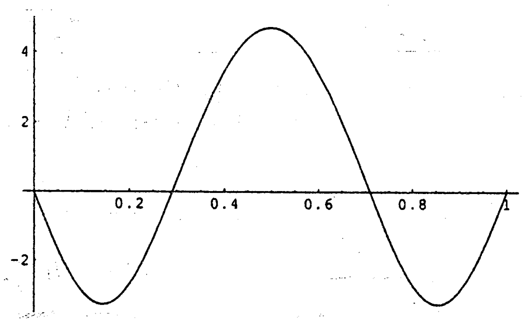
$$(4.6) \qquad \begin{cases} u_{xx} + u(a_1 - u - c_1 v) = 0 & \text{in} \quad (0,1), \\ v_{xx} + v(a_2 - c_2 u - v) = 0 & \text{in} \quad (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, & \\ u \geq 0, \quad v \geq 0 & \text{in} \quad (0,1). \end{cases}$$

Set

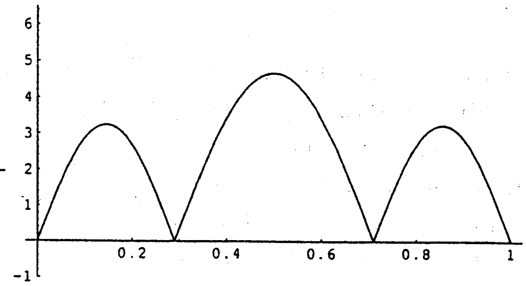
$$U = \frac{1}{a_1}u, \quad V = \frac{1}{a_2}v, \quad c = \frac{a_2 c_1}{a_1}, \quad d = \frac{a_1 c_2}{a_2};$$

then (4.6) is reduced to (3.5) for $\{U, V\}$ with $\mu = 1/a_1, \nu = 1/a_2$.

Numerical experiments have been done for $a_1 = 60, a_2 = 120$, which corresponds to $(\mu, \nu) = (1/60, 1/120) \in D_1^3$. In D_1^3, Lemma 4.1 implies $W = \{0, w_{0,\pm}, w_{1,\pm}\}$. The profile of $w_{2,-}$ is given in Figure 4 (A), the profile if the limit solution, i.e., $|w_{2,-}|$, is given in Figure 4 (B) and profiles of corresponding coexistence states are exhibited in Figure 5. Observe that (4.6) admits coexistence states which are very close to $|w_{2,-}|$ for sufficiently large interactions.



(A) Profile of $w_{2,-}$



(B) Profile of limit solution

FIGURE 4. sin-changing solution and limit solution

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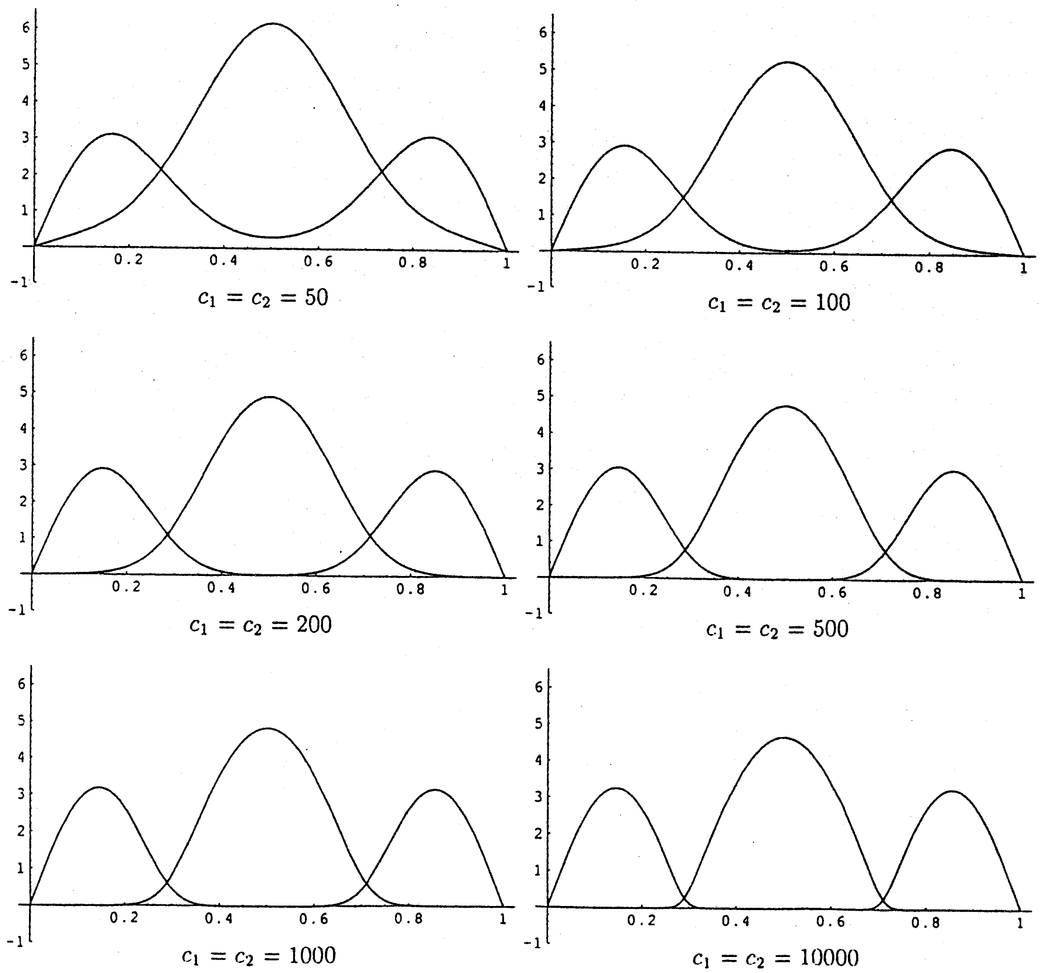

FIGURE 5. Profiles of coexistence states

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